Benha University Faculty of Engineering- Shoubra Eng. Mathematics & Physics Department Preparatory Year	Final Term Exam Date: December 16, 2017 Course: Mathematics 1 – A Duration: **3** hours
• The Exam consists of one page • Answer **All** Questions **(تخلفات)**	• No. of questions: 4 • Total Mark: 100

Question 1

Find **y`** from the following: [24]

(a) $y = 2x^4 + 4^x + x$ (b) $y = (x - \cos x)^6$ (c) $y = \log x - \ln(1 + \sin x)$

(d) $y = \tan^2 x + \sin x^3$ (e) $y^4 = x \ln y + \sin x$ (f) $y = t \cos t,\ x = t - \ln t$

Question 2

(a)Find the following limits: [8]

(i) $\lim\limits_{x \to 1} \dfrac{x^7 - 1}{1 - x^5}$ (ii) $\lim\limits_{x \to 0} \dfrac{\sin 2x}{2^x - 3^x}$ (iii) $\lim\limits_{x \to 0} \dfrac{x - \sin x}{x^3 + 2x}$ (iv) $\lim\limits_{x \to \infty} \dfrac{x^4 + 2x}{x + 3x^5}$

(b)Write the Maclurin's expansion of the function : $f(x) = (x + 1) \sin x$. [4]

(c)Sketch the curve of the function : $f(x) = x + \frac{1}{x}$ [4]

(d)State and verify Rolle's theorem for, $f(x) = (x - 1)^2$ in the interval [0, 2]. [4]

(e)Find the integrals: (i) $\int (x^3 + 3^x)\, dx$ (ii) $\int (1 - 2^x)^2\, dx$ (iii) $\int (1 + \cos 2x)\, dx$ [6]

Question 3

(a) Prove that : $1 + 2 + 3 + \cdots n = \frac{1}{2} n(n + 1)$ for $n \geq 1$. [6]

(b)Find the sum : $\sum\limits_{r=1}^{n} (4r^2 - 1)$ [6]

(c) Find the coefficient of x^{15} in the expansion : $(1 + x)^4 . (1 - x)^{-4}$ [6]

(d) Express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos \theta$ and $\sin \theta$. [6]

Question 4

(a) Resolve the fraction [6]

$$\frac{x^4 - 2x^3 + 3x^2 - x + 3}{x^3 - 2x^2 + 3x}$$

into its partial fractions.

(b)Write the first four terms in the expansion: $\sqrt[3]{8 - 2x}$. [6]

(c)Use the Gauss-Jordan algorithm to solve the linear system : [7]

$x_1 - 2x_2 + 2x_3 = 2,\ 3x_1 + x_2 - x_3 = 6,\ 2x_1 - 2x_2 - 3x_3 = -6$

(d)Show that $x = 2i$ is a zero to the polynomial [7]

$f(x) = x^4 - 5x^3 + 10x^2 - 20x + 24$ and then solve the equation f(x) = 0

Good Luck *Dr. Mohamed Eid* *Dr. Fathi Abdsallam*

Model Answer

Answer of Question 1

(a) $y` = 8x^3 + 4^x . \ln 4 + 1$

(b) $y` = 6(x - \cos x)^5 (1 + \sin x)$

(c) $y` = \frac{1}{\ln 10} . \frac{1}{x} - \frac{\cos x}{1+\sin x}$

(d) $y` = 2 \tan x . \sec^2 x + \cos x^3 . 3x^2$

(e) $4 y^3 . y` = \ln y + x . \frac{y`}{y} + \cos x$

(f) $y` = \frac{\cos t - t . \sin t}{1 - \frac{1}{t}}$

--*24-Marks*

Answer of Question 2

(a)(i) $\lim_{x \to 1} \frac{x^7 - 1}{1 - x^5} = \frac{0}{0} = -\frac{7}{5}$

(ii) $\lim_{x \to 0} \frac{\sin 2x}{2^x - 3^x} = \frac{0}{0} = \frac{2}{\ln(2/3)}$

(iii) $\lim_{x \to 0} \frac{x - \sin x}{x^3 + 2x} = \frac{0}{0} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2 + 2} = \frac{0}{2} = 0$

(iv) $\lim_{x \to \infty} \frac{x^4 + 2x}{x + 3x^5} = \frac{\infty}{\infty} = \lim_{x \to \infty} \frac{1 + \frac{2}{x^3}}{\frac{1}{x^3} + 3x} = \frac{1}{\infty} = 0$

--*8-Marks*

(b) $f(x) = (x + 1) \sin x = 0 + x + 2x^2 - \frac{1}{6} x^3$ ...

--*4-Marks*

(c) The domain of this function is R – {0}.

This function is odd.

The curve does not intersect y-axis.

At y = 0, we get the equation $x + \frac{1}{x} \neq 0$ which has no real roots. Then the curve does not intersect x-axis.

From $f`(x) = 1 - \frac{1}{x^2} = 0$, we get the equation $x^2 - 1 = 0$

which has the roots x = 1, – 1.

Since $f``(x) = \frac{2}{x^3}$.

Since $f``(1) = 2$ is positive, then (1, f(1)) = (1, 2) is minimum point.

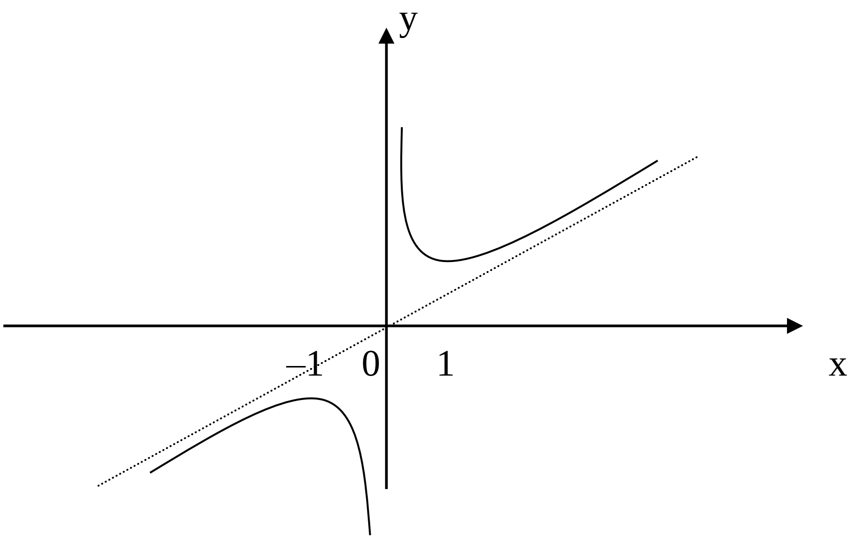
Since $f``(-1) = -2$ is negative, then (– 1, f(–1)) = (–1, – 2) is maximum point.

This function has no inflection points because $f``(x) = \frac{2}{x^3} \neq 0.$

This function has vertical asymptotic line x = 0 and has no horizontal asymptote because

$\lim_{x\to\infty} f(x) = \infty$.

It has inclined asymptotic line $y = x$.



The graph of f is shown in the Figure.

---*4-Marks*

(d)We see that $f(x)$, its derivative $f`(x) = 2(x-1)$ are continuous in the given interval [0, 2] and f(0) = f(2) =1.

Then $f`(c) = 2(c-1) = 0$. Hence c = 1.

---*4-Marks*

$$(e)(i) \int (x^3 + 3^x)\, dx = \frac{1}{4}x^4 - \frac{3^x}{\ln 3} + c$$

$$\text{(ii)} \int (1-2^x)^2\, dx = \int (1 - 2\, 2^x + 4^x) dx = x + 2\frac{2^x}{\ln 2} + \frac{4^x}{\ln 4} + c$$

---*6-Marks*

Dr. Mohamed Eid

<u>Answer of Question 3</u>

(a) Prove that $1+2+3+...+n = \frac{n(n+1)}{2}$ for any integer $n \geq 1$.

<u>Answer</u>

STEP 1: For n=1 ***P*(1)** is true, since 1 =1(1 + 1)/2.

STEP 2: Suppose $P(k)$ is true for some $k \geq 1$,

That is $$1+2+3+...+k=\frac{k(k+1)}{2}.$$

STEP 3: Prove that $P(k+1)$ is true for $n=k+1$,

That is $$1+2+3+...+k+(k+1)=\frac{(k+1)(k+2)}{2}.$$

We have $1+2+3+...+k+(k+1)=\dfrac{k(k+1)}{2}+(k+1)=\dfrac{(k+1)(k+2)}{2}$,

Which show that $p(k+1)$ is true

---6-*Marks*

(b) Find the sum $\sum_{r=1}^{n}(4r^2-1)$

$$\sum_{r=1}^{n}(4r^2-1)=\sum_{r=1}^{n}(2r-1)(2r+1)$$

$$u_r=(2r-1)(2r+1)=\frac{(2r-1)(2r+1)(2r+3)}{6}-\frac{(2r-3)(2r-1)(2r+1)}{6}$$

$$=f(r+1)-f(r)$$

$$S_n=\frac{(2n-1)(2n+1)(2n+3)}{6}+\frac{1}{2}$$

$$then\ \boxed{\sum_{r=1}^{n}(4r-1)^2=\frac{(2n-1)(2n+1)(2n+3)}{6}+\frac{1}{2}}$$

---6-*Marks*

(c) Find coefficient x^{15} in the expansion $(1+x)^4(1-x)^{-4}$.

$$(1+x)^4(1-x)^{-4}=\left(1+4x+6x^2+4x^3+x^4\right)\left(\sum_{r=0}^{\infty}{}^{-4}C_r(-x)^r\right)$$

$$=\left(1+4x+6x^2+4x^3+x^4\right)\left(\sum_{r=0}^{\infty}(-1)^r\,C_r^{4+r-1}(-1)^r x^r\right)$$

$$=(1+4x+6x^2+4x^3+x^4)\left(\sum_{r=0}^{\infty}C_r^{4+r-1}x^r\right)$$

$$=(1+4x+6x^2+4x^3+x^4)\left(C_0^3+C_1^4x+C_2^5x^2+...+C_n^{3+n}x^n+...\right)$$

Coefficient x^{15} is

$$C_{15}^{18}+4C_{14}^{17}+6C_{13}^{16}+4C_{12}^{15}+C_{11}^{14}$$

$$=C_{3}^{18}+4C_{3}^{17}+6C_{3}^{16}+4C_{3}^{15}+C_{3}^{14}$$

$$=\frac{18.17.16}{3.2.1}+4.\frac{17.16.15}{3.2.1}6.\frac{16.15.14}{3.2.1}+4.\frac{15.14.13}{3.2.1}+\frac{14.13.12}{3.2.1}=9080$$

--*6-Marks*

(d) Express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$

Using the equation

$$(\cos n\theta + i\sin n\theta) = (\cos\theta + i\sin\theta)^3 =$$

$$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$

$$= \cos^3\theta + 3i\cos^2\theta(\sin\theta) - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$$

Equating real and imaginary parts gives

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

--*6-Marks*

Answer of Question 4

(a) $\dfrac{x^4-2x^3+3x^2-x+3}{x^3-2x^2+3x}$

$$\frac{x^4-2x^3+3x^2-x+3}{x^3-2x^2+3x}=x+\frac{-x+3}{x(x^2-2x+3)}$$

$$\frac{-x+3}{x(x^2-2x+3)}=\frac{A}{x}+\frac{Bx+C}{x^2-2x+3}$$

--6-Marks

(b) $\sqrt[3]{8-2x}$

$$\sqrt[3]{8-2x}=2(1-\frac{x}{4})^{\frac{1}{3}}=2\left[1-\frac{1}{3}(\frac{x}{4})+\frac{1}{2!}(\frac{1}{3})(\frac{1}{3}-1)(\frac{x}{4})^2-\frac{1}{3!}(\frac{1}{3})(\frac{1}{3}-1)(\frac{1}{3}-2)(\frac{x}{4})^3+\ldots\right]$$

$$=2\left[1-\frac{x}{12}-\frac{x^2}{144}-\frac{5x^3}{5184}+\ldots\right]$$

--*6-Marks*

(c)Use the Gauss-Jordan algorithm to solve the linear system :

$x_1-2x_2+2x_3=2,\ 3x_1+x_2-x_3=6,\ 2x_1-2x_2-3x_3=-6$

Solution

The augmented matrix of the system is

$$\left(\begin{array}{ccc|c} \mathbf{1} & \mathbf{-2} & \mathbf{2} & \mathbf{2} \\ \mathbf{3} & \mathbf{1} & \mathbf{-1} & \mathbf{6} \\ \mathbf{2} & \mathbf{-2} & \mathbf{-3} & \mathbf{-6} \end{array}\right)$$

Applying Gauss-Jordan algorithm to this matrix yields

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 7 & -7 & 0 \\ 0 & 2 & -7 & -10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 2 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -7 & -10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -5 & -10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \text{the system has unique solution} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

--*7-Marks*

(d)Show that $x = 2i$ is a zero to the polynomial

$f(x) = x^4 - 5x^3 + 10x^2 - 20x + 24$ and then solve the equation f(x) = 0.

<u>Solution</u>

David the given polynomial by the linear expression $(x - 2i)$ and we found that the remainder

$R = f(x - 2i) = 0$

This means that the value $2i$ is a zero of $f(x)$ and $2i$ is a root for $f(x) = 0$, then $-2i$ is another root and $f(x) = (x - 2i)(x + 2i)Q(x) = (x^2 + 4)Q(x)$

The polynomial $Q(x)$ is obtained by synthetic division as follows

2i	1	-5	10	-20	24
			-4-		
		2i	10i	20+12i	-24
			6-		
-2i	1	-5+2i	10i	12i	0
		-2i	10i	12i	
	1	-5	6	0	

Then

$f(x) = (x^2 + 4)(x^2 - 5x + 6) = (x^2 + 4)(x - 2)(x - 3)$

The roots are $\pm 2i$, 2, 3

--*7-Marks*

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